**ISI B. Math.**
**Physics I**
**Final Exam**
**Total Marks: 100**

Answer any five questions. All questions carry equal marks.

1. Suppose that the sun were surrounded by a dust cloud of uniform density $\rho$ which extended at least as far as the orbital radius of the earth. The effect of the dust cloud is to modify the gravitational force experienced by the earth, so that the potential energy of the earth (neglecting the effect of the other planets)

$$U(r) = -\frac{GMm}{r} + \frac{1}{2}kr^2$$

where $M$ is the mass of the sun, $m$ is the mass of the earth, $G$ is the gravitational constant, and $k = 4\pi\rho mG/3$.
(a) From the potential, find the force $\mathbf{F}$ acting upon the earth.
(b) Make a careful sketch of the effective potential $U_{eff}(r)$. On your sketch, indicate (i)energy $E_0$ and radius $r_0$ of a circular orbit, and (ii) the energy $E_1$ and turning points $r_1$ and $r_2$ of an orbit which is not circular.
(c) Assume that the earth is in a circular orbit of radius $r_0$ about the sun. Derive the equation satisfied by $r_0$ in terms of the angular momentum $l$ and the constants $m, M, G$ and $k$. You need not solve the equation.
(d) Find the frequency of small oscillations $\omega_r$ about the circular orbit of radius $r_0$. You should find that your result can be written as

$$\omega_r = \sqrt{\omega_0^2 + \frac{3k}{m}}$$

(e) Finally, by assuming $k$ is small, show that the precession frequency $\omega_p$ for a nearly circular orbit $= \frac{3k}{2m\omega_0}$, recalling that the precession frequency is given by the difference between the small oscillation frequency and the orbital frequency of the earth.

1. Consider a charged particle of mass $m$ and charge $q$ entering a uniform constant magnetic field $\mathbf{B} = B_0\hat{\mathbf{k}}$. The force on
the charged particle is given by

$$\mathbf{F} = \frac{q}{c}(\mathbf{v} \times \mathbf{B})$$

where $c$ is the speed of light.
a) Show that the kinetic energy of the particle is a constant of motion.
b) Given that at time $t = 0$, the particle starts from the origin with $\dot{x} = 0, \dot{y} = \dot{y}_0, \dot{z} = \dot{z}_0$, find the subsequent motion of the particle and make a rough sketch of its trajectory. How will the trajectory be affected by increasing the magnetic field?

3. A circular disc of radius $a$ lies in the $x - y$ plane with its centre at the origin. The half of the disk above the $x-$ axis has a density $\sigma$ per unit area, and the half below the $x-$ axis has a density $2\sigma$. Find the centre of mass $G$, and the moment of inertia about the $x, y$ and $z$ axes and about parallel axes through $G$. Make as much use of labour saving theorems as possible.

4. (a) Use the technique of calculus of variations to show that the curve of shortest length between two fixed points in a plane is a straight line.
(b) Consider a massless, frictionless pulley with a mass $M_1$ hanging at one end and a mass $M_2$ hanging from the other end as shown in the figure. Write down the Lagrangian $L(x, \dot{x})$ for the system and find the acceleration of the masses by applying the Euler-Lagrange equation to the Lagrangian.
(c) Consider a free particle of mass $m$ moving in one dimension with respect to a certain inertial frame $K$. Now consider another inertial frame $K'$ moving with a constant velocity $v$ with respect to $K$. Show that the Lagrangian $L'$ in the frame $K'$ differs from the Lagrangian $L$ in $K$ by a total time derivative of a function of coordinates and time.

5. (a) Consider a rectangular block of elastic material that is stretched along its length such that its length $l$ is increased by $\Delta l$. During this process, appropriate forces are applied to the block to ensure that there is no contraction of the block in the directions perpendicular to the direction in which

the block is stretched. Show that

$$\frac{F}{A} = \frac{1-\sigma}{(1+\sigma)(1-2\sigma)} Y \frac{\Delta l}{l}$$

where $F$ is the magnitude of the stretching force, $A$ is the cross-sectional area perpendicular to the force, $Y$ is the Young's modulus and $\sigma$ is Poisson's ratio for the material. Is the factor multiplying the strain greater than or less than 1 ? Explain.

(b) If the torsional rigidity (torque per unit twist) of a wire is given by $c$, and a body of moment of inertia $I$ suspended from the wire undergoes small torsional oscillations, show that the period of small oscillations $T$ is given by $2\pi\sqrt{\frac{I}{c}}$. Consider the experimental arrangement shown in the figure . One end of a wire is fixed vertically to a rigid support and the other end is fixed to the centre of a metallic disc. Two known masses ($m$ each) are placed symmetrically along the diameter at a given distance and the disc + mass system is made to undergo small torsional oscillations. Show that the rigidity modulus $\mu$ is given by

$$\mu = \frac{16\pi m l(d_2^2 - d_1^2)}{a^4(T_2^2 - T_1^2)}$$

where $T_2$ and $T_1$ are the time periods of oscillation when the masses are placed at distances $d_2$ and $d_1$ respectively. $l$ and $a$ are the length and radius of the wire respectively and $c = \frac{\pi\mu a^4}{2l}$. This is an experimental arrangement to determine the modulus of rigidity of a wire.

6. a) Show that the flow defined by the velocity field

$$(2t + 2x + 2y)\hat{\mathbf{i}} + (t - y - z)\hat{\mathbf{j}} + (t + x - z)\hat{\mathbf{k}}$$

is incompressible.

b) Euler's equation for an ideal fluid with velocity $\mathbf{v}$, pressure $p$ and density $\rho$ flowing under the influence of an external force with potential $\phi$ is given by

$$\frac{\partial v}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} = -\frac{\nabla p}{\rho} - \nabla\phi$$

Show that this reduces to

$$\frac{\partial \mathbf{\Omega}}{\partial t} + \nabla \times (\mathbf{\Omega} \times \mathbf{v}) = \mathbf{0}$$

for incompressible flow, where the vorticity $\mathbf{\Omega} = \nabla \times \mathbf{v}$

(c) Suppose we have water flowing out of a hole at the bottom of a tank as shown in the figure. Find the velocity $v_{out}$ of the water flowing out of the hole in terms of the depth $h$ of the hole using Bernoulli's principle.Assume the diameter of the tank is so large that we can neglect the drop in the liquid level.

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Some formulas that may be useful:

$$(\mathbf{A} \cdot \nabla)\mathbf{A} = (\nabla \times \mathbf{A}) \times \mathbf{A} + \frac{1}{2}\nabla(\mathbf{A} \cdot \mathbf{A})$$

In spherical polar coordinates:

$$\nabla = \hat{\mathbf{r}}\frac{\partial}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial \theta} + \hat{\phi}\frac{1}{r \sin\theta}\frac{\partial}{\partial \phi}$$